O globalnych własnościach funkcji półciągłych z dołu minoryzowanych przez funkcje kwadratowe

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X - dowolny zbiór, Φ: $\varphi : X \to \mathbb{R}$

DEFINITION 1. The set $\text{supp}(f) := \{\varphi \in \Phi : \varphi \le f\}$ is called the *support* of f.

$$\varphi \le f \iff \varphi(x) \le f(x) \quad \forall x \in X$$

DEFINITION 2. A function f is called Φ-*convex* if

$$f(x) = \sup\{\varphi(x) \ : \ \varphi \in \text{supp}(f)\} \quad \forall \, x \in X.$$

X - przestrzeń Hilberta, X^* - dualna, $\langle \cdot , \cdot \rangle$ - iloczyn skalarny

$$\Phi_{lsc} := \{\varphi : X \to \mathbb{R}, \ \varphi(x) = -a\|x\|^2 + \langle v, x\rangle + c, \ \ x \in X, \ v \in X^*, \ a \ge 0, \ c \in \mathbb{R}\}.$$

PROPOSITION 1. *Let $f : X \to \bar{\mathbb{R}}$ be a proper function. f is Φ_{lsc}-convex on X if and only if f is lower semicontinuous on X and minorized by a function from the class Φ_{lsc}.*

$$\mathrm{dom}(f) = \{x \in X : f(x) < +\infty\} \neq \emptyset$$

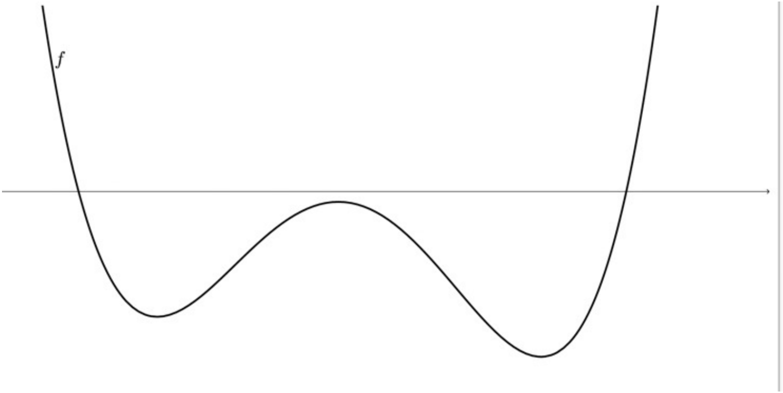
DEFINITION 3. A function $f : X \to \bar{\mathbb{R}}$ is called *paraconvex* on X if there exists $C > 0$ such that for all $x, y \in X$ and $t \in [0,1]$ the following inequality holds

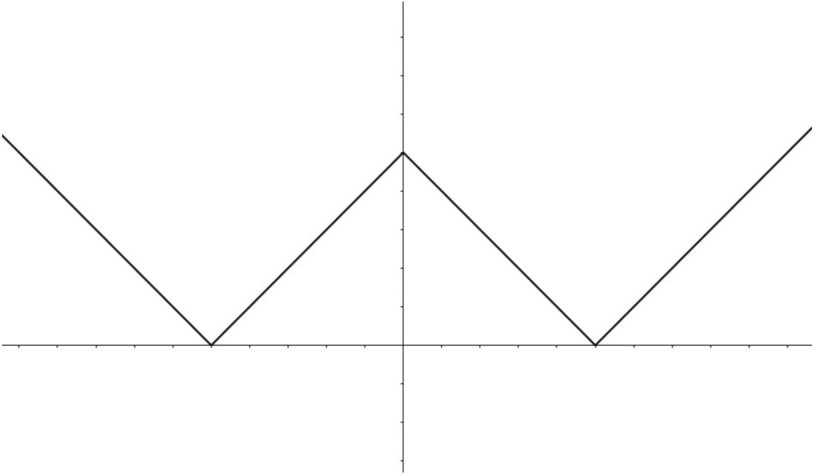
$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + C\|x-y\|^2. \tag{1}$$

DEFINITION 4. A function $f : X \to \bar{\mathbb{R}}$ is *weakly convex* on X if there exists $c > 0$ such that the function $f(x) + c\|x\|^2$ is convex.

PROPOSITION 2. *Let $f : X \to \bar{\mathbb{R}}$ be a proper function. f is weakly convex on X if and only if f is paraconvex on X.*

PROPOSITION 3. *Let $f : X \to \bar{\mathbb{R}}$ be a proper lower semicontinuous function. If f is paraconvex on X then f is Φ_{lsc}-convex on X.*

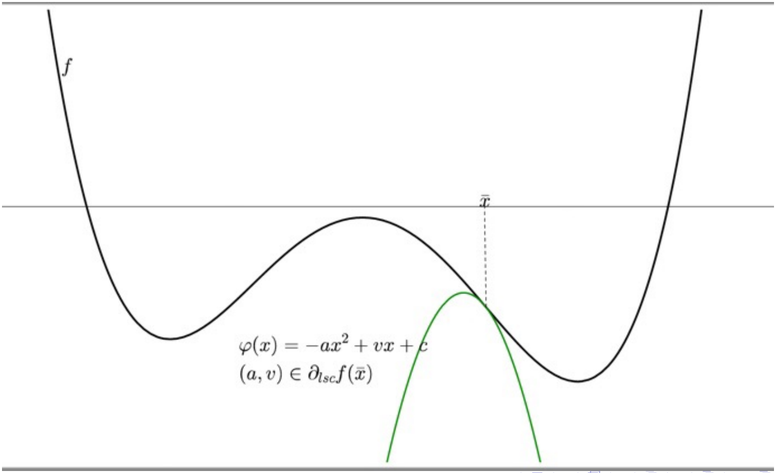


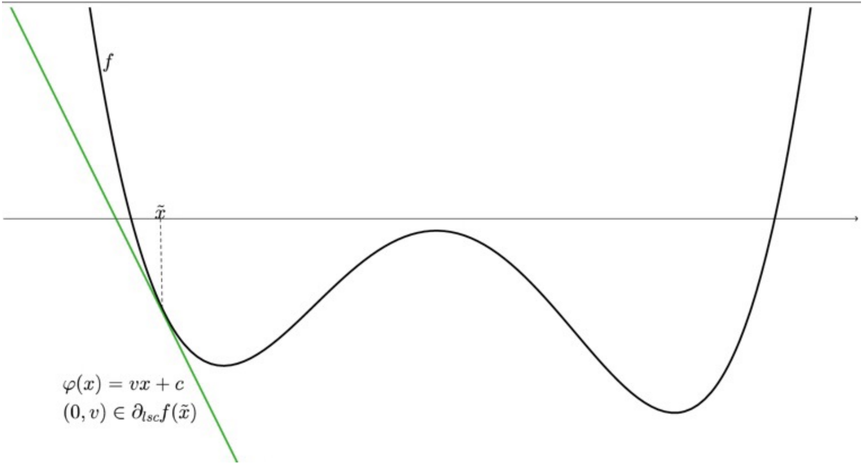


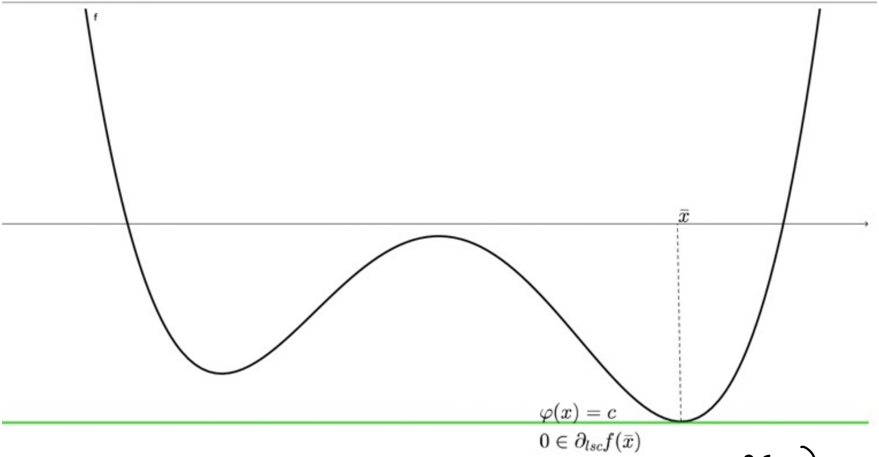
$\partial_{lsc} f(\bar{x})$ zbiór wszystkich Φ_{lsc} subgradientów

DEFINITION 5. Let $f : X \to \bar{\mathbb{R}}$ be a proper function. An element $(a, v) \in \mathbb{R}_+ \times X^*$ is called a Φ_{lsc}-*subgradient* of f at $\bar{x} \in \mathrm{dom}(f)$, if the following inequality holds

$$f(x) - f(\bar{x}) \geq \langle v, x - \bar{x} \rangle - a\|x\|^2 + a\|\bar{x}\|^2, \qquad \forall\, x \in X. \tag{2}$$




f
$\tilde{x}$
$\varphi(x) = vx + c$
$(0, v) \in \partial_{lsc} f(\tilde{x})$



$(0,0) \in \partial_{\lambda sc} f(\bar{x})$

DEFINITION 6. Let $f : X \to \bar{\mathbb{R}}$ be a proper function. An element $(a, v) \in \mathbb{R}_+ \times X^*$ is called a *local* Φ_{lsc}*-subgradient* of f at $\bar{x} \in \mathrm{dom}(f)$, if there exists $\delta > 0$ such that, the following inequality holds

$$f(x) - f(\bar{x}) \geq \langle v, x - \bar{x}\rangle - a\|x\|^2 + a\|\bar{x}\|^2, \qquad \forall\ \|x - \bar{x}\| < \delta. \tag{3}$$

The set of all local Φ_{lsc}-subgradients of f at $\bar{x}$ is denoted by $\partial^{\mathrm{loc}}_{lsc} f(\bar{x})$.

$\forall x \in X : \|x - \bar{x}\| < \delta$

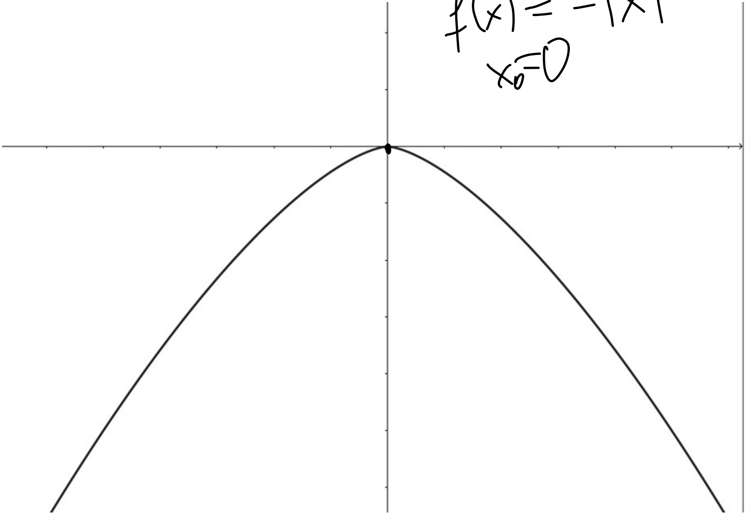
PROPOSITION 4. *Let $f : X \to \bar{\mathbb{R}}$ be a proper Φ_{lsc}-convex function and $\bar{x} \in dom(f)$. If $(a, v) \in \partial^{loc}_{lsc} f(\bar{x})$ then there exists $\bar{a} \geq 0$ such that $(\bar{a}, v - 2a\bar{x} + 2\bar{a}\bar{x}) \in \partial_{lsc} f(\bar{x})$.*

$$\bar{a} = \max\left\{a, \frac{f(\bar{x}) + \rho\|\bar{x}\|^2 - c}{\delta^2} + \rho\left(1 + \frac{2\|\bar{x}\|}{\delta}\right)\right\}$$

$\exists\ \rho, c, v \qquad f(x) \geq -\rho\|x\|^2 + \langle v, x\rangle + c$

$$f(x) = -|x|^{\frac{3}{2}}$$

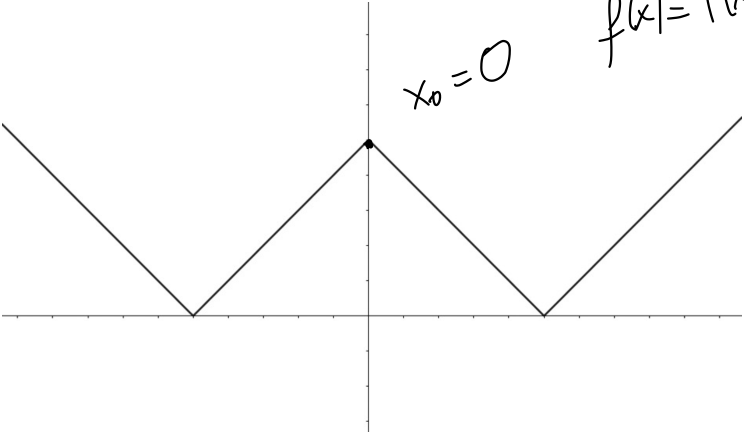
$$x_0 = 0$$



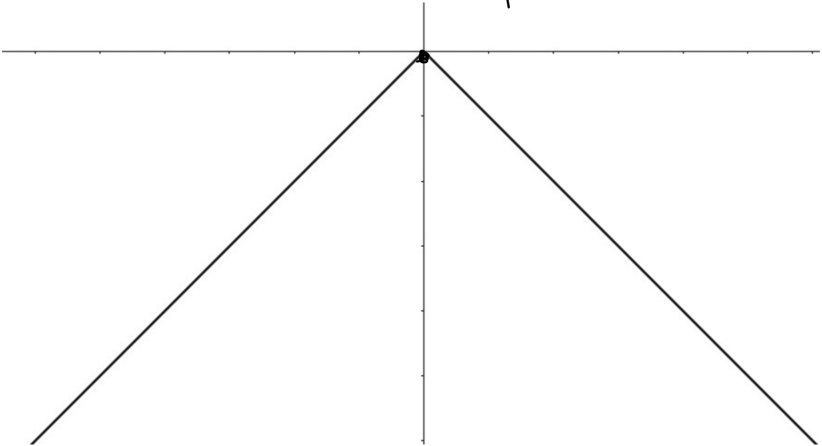
PROPOSITION 5. *Let $f : X \to \bar{\mathbb{R}}$ be a proper l.s.c. paraconvex function, then for every $x \in \operatorname{int}\operatorname{dom}(f)$ the set $\partial_{lsc} f(x)$ is nonempty.*

$f(x) = ||x| - 1|$

$x_0 = 0$



$f(x) = -|x|$



Proposition 6. *Let $f : X \to \bar{\mathbb{R}}$ be a proper function and $U \subset X$ be an open convex set such that $U \subset \mathrm{dom}(f)$. If there exists $a \geq 0$, such that $(a, v_{\bar{x}}) \in \partial_{lsc} f(\bar{x})$ for every $\bar{x} \in U$, then f is paraconvex on U.*

Proposition 7. *Let $f : X \to R$ be a proper Φ_{lsc}-convex function. If f is $C^{1,1}$ around $\bar{x} \in \mathrm{dom}(f)$, then there exists $\delta > 0$ such that, for every $y \in B(\delta, \bar{x})$, the set $\partial_{lsc} f(y)$ is nonempty.*

Proposition 8. *Let $f : X \to R$ be a proper Φ_{lsc}-convex function and U be an open subset of X. If $f \in C^2(U)$, then for every $x \in U$, the set $\partial_{lsc} f(x)$ is nonempty.*

$\varphi_1, \varphi_2 \in \Phi_{lsc}$

Funkcje $\varphi_1, \varphi_2 : X \to \mathbb{R}$ mają własność przecięcia na poziomie α jeżeli

$$[\varphi_1 < \alpha] \cap [\varphi_2 < \alpha] = \emptyset$$

$$[\varphi_1 < \alpha] = \{x \in X : \varphi_1(x) < \alpha\}$$

Definition 7. Let $f, g : X \to \bar{\mathbb{R}}$ be Φ_{lsc}-convex functions, $x_1 \in \mathrm{dom}(f)$, $x_2 \in \mathrm{dom}(g)$ and ~~[illegible]~~ We say that f and g satisfy the *zero subgradient condition* at (x_1, x_2), ~~[illegible]~~ if

$$0 \in \mathrm{co}(\partial_{lsc} f(x_1) \cup \partial_{lsc} g(x_2)),$$

where $\mathrm{co}(\cdot)$ is a standard convex hull of a set.

Prop. $f, g : X \to \bar{\mathbb{R}}$ są Φ_{lsc} wypukłe. $\alpha \in \mathbb{R}$. Załóżmy, że $\bar{x} \in \mathrm{dom}(f) \cap \mathrm{dom}(g)$
$\bar{x} \in [f \geq \alpha] \cap [g \geq \alpha]$.
Jeżeli funkcji f i g spełniają warunek zerowego subgradientu tzn. $0 \in \mathrm{co}(\partial_{lsc} f(\bar{x}) \cup \partial_{lsc} g(\bar{x}))$

to $\exists\, \varphi_1 \in \operatorname{supp}(f)$ i $\varphi_2 \in \operatorname{supp}(g)$ takie, że φ_1 i φ_2 mają własność przecięcia na poziomie α.

$[\varphi_1 < \alpha] \cap [\varphi_2 < \alpha] = \emptyset$

S-Lemma

Funkcje $\varphi_1, \varphi_2 \in$ ~~...~~ mają własność przecięcia na poziomie α

$\Longleftrightarrow \exists\, t_0 \in [0,1]$

$$t_0 \varphi_1(x) + (1 - t_0)\varphi_2(x) \geq \alpha \quad \forall x \in X.$$

$[\varphi_1 < \alpha] \cap [\varphi_2 < \alpha] \Leftarrow \varphi_1 < \alpha \Rightarrow \varphi_2 \geq \alpha .$

$$f(x) = \begin{cases} -1 & x \in (-\infty, -1) \\ x^3 & x \geq -1 \end{cases}$$